Section 8.1: The Taylor Polynomial Approximation

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The Taylor Polynomial Approximation to a GIVEN Function (Story)

- What is a polynomial?

- Why do we like polynomials?

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- Suppose we want to calculate $\sin(0.03)$. Is $f(x) = \sin x$ a degree 1 polynomial?

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The Taylor Polynomial Approximation to a GIVEN Function (Story)

- Is $f(x) = \sin x$ a degree 3 polynomial?

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<u>The Taylor Polynomial Approximation to a GIVEN Function (Story)</u>

- What happens if we keep doing this? Is $f(x) = \sin x$ a degree n polynomial?

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The Taylor Polynomial Approximation to a GIVEN Function (Story)

- Taylor series generated by a given function f

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

Taylor series generated by f
centered at 0

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i$$

Taylor series generated by f
centered at x_0

$$f(x) \approx \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i$$

Taylor polynomial approximation to f
centered at 0

$$f(x) \approx \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i$$

Taylor polynomial approximation to f
centered at x_0

Some Problems

- f needs to be infinitely differentiable in order to generate a Taylor series, otherwise the best we can do is get a polynomial approximation
- Does the Taylor series generated by f always converge to f?

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The Taylor Polynomial Approximation to a GIVEN Function (Story)

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Some Problems

- f needs to be infinitely differentiable in order to generate a Taylor series, otherwise the best we can do is get a polynomial approximation
- Does the Taylor series generated by f always converge to f?

In this section we will assume that…

- Taylor polynomials are "good" approximations to the given function f
- The Taylor series generated by f "always" converges to f

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The Taylor Polynomial Approximation to a GIVEN Function

Example 1 Determine the fourth-degree Taylor polynomials matching the functions e^x, $\cos x$, and $\sin x$ at $x_0 = 2$.

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The Taylor Polynomial Approximation to the Solution to an IVP (Story)

The goal of this section is…

- Given an IVP, instead of finding the exact solution, find a Taylor polynomial approximation to the solution (i.e. find an approximate solution)
- This time around, instead of finding a Taylor polynomial approximation to a given function f, we are going to find a Taylor polynomial approximation to the solution to the IVP $y(t)$
- To do this, we need to know $y(0)$, $y'(0)$, $y''(0)$, $y'''(0)$, … and so on. Some of these are given initial conditions and some we find using the differential equation

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The Taylor Polynomial Approximation to the Solution to an IVP

Example 2 Find the first few Taylor polynomials approximating the solution around $x_0 = 0$ of the initial value problem

$$y'' = 3y' + x^2 y; \qquad y(0) = 10, \qquad y'(0) = 5.$$

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The Taylor Polynomial Approximation to the Solution to an IVP

Example 2 Find the first few Taylor polynomials approximating the solution around $x_0 = 0$ of the initial value problem

$$y'' = 3y' + x^2 y\,; \qquad y(0) = 10\,, \qquad y'(0) = 5\,.$$

It is of interest to note that if the original equation in Example 2 were replaced by $y'' = 3y' + x^{1/3}y$, the third derivative would look like $y''' = 3y'' + y/(3x^{2/3}) + x^{1/3}y'$, and $y'''(0)$ would not exist. Only Taylor polynomials of degree 0 through 2 can be constructed for the solution to this problem.

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The Taylor Polynomial Approximation to the Solution to an IVP

Example 3 Determine the Taylor polynomial of degree 3 for the solution to the initial value problem

$$y' = \frac{1}{x+y+1}, \qquad y(0) = 0. \tag{4}$$

Section 8.1: The Taylor Polynomial Approximation

<u>Question</u>: When does the Taylor series generated by f converge to f?

$$\varepsilon_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

Figure 8.1 on page 422 and equation (6) suggest that one might control the error in the Taylor polynomial approximation by increasing the degree n of the polynomial (i.e., taking more terms), thereby increasing the factor $(n+1)!$ in the denominator. This possibility is limited, of course, by the number of times f can be differentiated. In Example 2, for instance, the solution did not have a fifth derivative at $x_0 = 0$ ($f^{(5)}(0)$ is "infinite"). Thus, we could not construct $p_5(x)$, nor could we conclude anything about the accuracy of $p_4(x)$ from the Lagrange formula.

However, for Example 3 we could, in theory, compute *every* derivative of the solution $y(x)$ at $x_0 = 0$, and speculate on the *convergence* of the Taylor series

$$\sum_{j=0}^{\infty} \frac{y^{(j)}(x_0)}{j!}(x-x_0)^j = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{y^{(j)}(x_0)}{j!}(x-x_0)^j$$

to the solution $y(x)$. Now for nonlinear equations such as (4), the factor $f^{(n+1)}(\xi)$ in the Lagrange error formula may grow too rapidly with n, and the convergence can be thwarted. But if the differential equation is linear and its coefficients and nonhomogeneous term enjoy a feature known as *analyticity*, our wish is granted; the error does indeed diminish to zero as the degree n goes to infinity, and the sequence of Taylor polynomials can be guaranteed to converge to the actual solution on a certain (known) interval.